

Asymptotic study of the initial value problem to a standard one pressure model of multifluid flows in nondivergence form

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Abstract

We construct families of approximate solutions to the initial value problem and provide complete mathematical proofs that they tend to satisfy the standard system of isothermal one pressure two-fluid flows in 1-D when the data are L^1 in densities and L^∞ in velocities. To this end, we use a method that reduces this system of PDEs to a family of systems of four ODEs in Banach spaces whose smooth solutions are these approximate solutions. This method is constructive: using standard numerical methods for ODEs one can easily and accurately compute these approximate solutions which, therefore, from the mathematical proof, can serve for comparison with numerical schemes. One observes agreement with previously known solutions from scientific computing [S. Evje, T. Flatten. Hybrid Flux-splitting Schemes for a common two fluid model. J. Comput. Physics 192, 2003, p. 175-210]. We show that one recovers the solutions of these authors (exactly in one case, with a slight difference in another case). Then we propose an efficient numerical scheme for the original system of two-fluid flows and show it gives back exactly the same results as the theoretical solutions obtained above.

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1. Introduction.

We study a basic model used to describe mathematically a mixture of two immiscible fluids in the isothermal case and without transfer of momentum between the two fluids, [14] p. 179, [7] p. 465,

$$\frac{\partial}{\partial t}(\rho_1 \alpha_1) + \frac{\partial}{\partial x}(\rho_1 \alpha_1 u_1) = 0, \quad (1)$$

$$\frac{\partial}{\partial t}(\rho_2 \alpha_2) + \frac{\partial}{\partial x}(\rho_2 \alpha_2 u_2) = 0, \quad (2)$$

$$\frac{\partial}{\partial t}(\rho_1 \alpha_1 u_1) + \frac{\partial}{\partial x}(\rho_1 \alpha_1 (u_1)^2) + \frac{\partial}{\partial x}((p_1 - p_1^{int}) \alpha_1) + \alpha_1 \frac{\partial}{\partial x}(p_1^{int}) = g \alpha_1 \rho_1, \quad (3)$$

$$\frac{\partial}{\partial t}(\rho_2 \alpha_2 u_2) + \frac{\partial}{\partial x}(\rho_2 \alpha_2 (u_2)^2) + \frac{\partial}{\partial x}((p_2 - p_2^{int}) \alpha_2) + \alpha_2 \frac{\partial}{\partial x}(p_2^{int}) = g \alpha_2 \rho_2, \quad (4)$$

$$\alpha_1 + \alpha_2 = 1, \quad (5)$$

$$p_1 = K_1 \rho_1 - b_1, \quad p_2 = K_2 \rho_2 - b_2, \quad (6)$$

where the two fluids are denoted by the indices 1 and 2, for instance mixture of oil and natural gas in extraction tubes of oil exploitation [2]. The physical variables are the densities $\rho_i(x, t)$, the velocities $u_i(x, t)$, the volumic proportions $\alpha_i(x, t)$, the pressures $p_i(x, t)$, the phasic pressures $p_i^{int}(x, t)$ at the interface, $i = 1, 2$, and g is the component of the gravitational acceleration in the direction of the tube. Equations (6) are the state laws stated in [14] p. 179; it is assumed $b_1 - b_2 > 0$, $K_1 > 0$ and $K_2 > 0$. Equations (1) and (2) are the continuity equations for each fluid: they express mass conservation. Equations (3) and (4) are the Euler equations for each fluid: they express momentum conservation. A natural assumption is to state the equality of the four pressures p_i and p_i^{int} , $i = 1, 2$. This simplest assumption of equal pressure leads to a nonhyperbolic model, called the equal pressure model [13] p.677, [20] p. 2589, [23] p. 287, [24] p. 372-373 that we study in this paper.

We construct families of differentiable functions $S(x, t, \epsilon)$ that, when plugged into the equal pressure model, tend asymptotically to satisfy it when $\epsilon \rightarrow 0$. We prove that these families of functions are weak asymptotic methods. The concept of weak asymptotic method and its relevance has been put in

evidence by many authors [1, 8, 9, 21, 22] by explicit calculations and by reduction of the problem of description of nonlinear waves interaction to the resolution of systems of ordinary differential equations, as a continuation of Maslov's theory. In other words our families of functions tend to satisfy the system modulo a remainder that tends to 0 when $\epsilon \rightarrow 0$. To construct these families we use a method which consists in solving a system of four ordinary differential equations in a Banach space whose solutions are the approximate solutions of the one pressure model. This method allows us to compute the solutions with standard convergent numerical schemes for ODEs, thus permitting comparison with existing numerical solutions of the equal pressure model obtained in scientific computing. We observe the approximate solutions we obtain agree with the results presented in [14], with a small difference in one case which diminishes in presence of the pressure correction, which can be considered as a mathematical justification of these numerical results. The system (1-6) is in nondivergence form, i.e. the derivatives cannot be transferred to test functions because of the terms $\alpha_i \frac{\partial p_i^{int}}{\partial x}$ in (3, 4). Therefore the study of the solutions of this system in presence of shock waves is problematic and we use a family of approximate solutions that are classical differentiable functions which permits at the limit to obtain "exact solutions" that are irregular functions such as discontinuous functions. In this way the weak asymptotic methods presented here can be a tool for mathematical and numerical investigations of discontinuous solutions despite this system is in nondivergence form. Various systems in divergence form have been obtained by replacing (10,11) below by their sum, and then by introducing a new equation [10, 11, 12, 15, 17, 18, 19].

From a physical viewpoint the equations of fluid dynamics are marred with some imprecision since they do not take into account some minor effects and the molecular structure of matter. It is natural to expect these equations and their imprecision should be stated in the sense of distributions in the space variables. Weak asymptotic methods provide approximate solutions that enter into this imprecision for $\epsilon > 0$ small enough. Therefore they could be considered as some convenient way to approximate possible solutions to the equations of physics. In the absence of a uniqueness result of a privileged family of weak asymptotic methods (all giving same results) that should represent physics in a given physical situation, we have to content to check numerically that the weak asymptotic methods we present give the known solutions at the limit $\epsilon \rightarrow 0$.

2. Simplified statement of the system.

In order to simplify the study of the system (1-6) with the equal pressure assumption we transform it into a system of four equations with four unknown functions by changes of unknown and algebraic calculations. We set

$$r_1 = \rho_1 \alpha_1, r_2 = \rho_2 \alpha_2, \alpha = \alpha_1. \quad (7)$$

Then (1-5) with equal pressures has the form

$$\frac{\partial}{\partial t}(r_1) + \frac{\partial}{\partial x}(r_1 u_1) = 0, \quad (8)$$

$$\frac{\partial}{\partial t}(r_2) + \frac{\partial}{\partial x}(r_2 u_2) = 0 \quad (9)$$

$$\frac{\partial}{\partial t}(r_1 u_1) + \frac{\partial}{\partial x}(r_1 (u_1)^2) + \alpha \frac{\partial}{\partial x} p = g r_1, \quad (10)$$

$$\frac{\partial}{\partial t}(r_2 u_2) + \frac{\partial}{\partial x}(r_2 (u_2)^2) + (1 - \alpha) \frac{\partial}{\partial x} p = g r_2, \quad (11)$$

and the two state laws (6) are left unchanged. The 6 unknowns are now $r_1, r_2, u_1, u_2, \alpha$ and p . Then we transform the equations in a way which will be more convenient to construct the weak asymptotic method since we will have only the four unknown functions r_1, r_2, u_1 and u_2 .

- From (6), $p = K_1 \rho_1 - b_1 = K_2 \rho_2 - b_2$ implies

$$\rho_2 = \frac{-b_1 + b_2 + K_1 \rho_1}{K_2}. \quad (12)$$

Note that this calculation is linear so it can be done rigorously even in presence of shock waves.

- *Calculation of α in function of r_1 and r_2 .* We multiply the equality $p = K_1 \frac{r_1}{\alpha} - b_1 = K_2 \frac{r_2}{1-\alpha} - b_2$ by $\alpha(1-\alpha)$ to obtain (13) below: this is a nonlinear calculation. In the case of discontinuous solutions it is well known such nonlinear calculations usually change the solutions. This formal calculation is usual for this system and the observation of the numerical results in section 6, observation 3, shows a posteriori that this nonlinear calculation giving the formula (13) is justified. No unjustified nonlinear calculations are done after (13). This calculation gives

$$(1 - \alpha) K_1 r_1 - b_1 \alpha (1 - \alpha) = \alpha K_2 r_2 - b_2 \alpha (1 - \alpha), \quad (13)$$

i.e. $F(\alpha) = 0$, setting

$$F(X) = X^2(b_1 - b_2) + X(-K_1 r_1 - b_1 - K_2 r_2 + b_2) + K_1 r_1. \quad (14)$$

One has $F(0) = K_1 r_1 > 0$ and $F(1) = -K_2 r_2 < 0$ which implies that F has one and only one root $\alpha \in]0, 1[$ in the case $r_1 > 0$ and $r_2 > 0$ i.e. in absence of void regions in each fluid. In this case, since $F(1) < 0$ and since it is assumed $b_1 - b_2 > 0$, the second root is > 1 . Therefore the discriminant $\Delta = (K_1 r_1 + K_2 r_2 + b_1 - b_2)^2 - 4(b_1 - b_2)K_1 r_1$ is > 0 and the solution $\alpha \in]0, 1[$ is given by

$$\alpha = \frac{K_1 r_1 + K_2 r_2 + b_1 - b_2 - (\Delta)^{\frac{1}{2}}}{2(b_1 - b_2)}. \quad (15)$$

- The following result will be used below

$$0 \leq \frac{K_1 r_1}{b_1 - b_2 + K_1 r_1 + K_2 r_2} \leq \alpha \leq 1. \quad (16)$$

Proof. From (14), $F(X) \geq -X(b_1 - b_2 + K_1 r_1 + K_2 r_2) + K_1 r_1$ since $b_1 - b_2 > 0$; therefore $F(\frac{K_1 r_1}{b_1 - b_2 + K_1 r_1 + K_2 r_2}) \geq 0$, hence the result since $F(\alpha) = 0$ and $F(1) \leq 0$. \square

Finally we can eliminate α from (10, 11) and we obtain the following statement of the system: first the continuity equations

$$\frac{\partial}{\partial t}(r_1) + \frac{\partial}{\partial x}(r_1 u_1) = 0, \quad (17)$$

$$\frac{\partial}{\partial t}(r_2) + \frac{\partial}{\partial x}(r_2 u_2) = 0, \quad (18)$$

then the Euler equations in the form

$$\frac{\partial}{\partial t}(r_1 u_1) + \frac{\partial}{\partial x}(r_1 (u_1)^2) + r_1 \frac{\partial}{\partial x} \Phi_1 = g r_1, \quad (19)$$

$$\frac{\partial}{\partial t}(r_2 u_2) + \frac{\partial}{\partial x}(r_2 (u_2)^2) + r_2 \frac{\partial}{\partial x} \Phi_2 = g r_2, \quad (20)$$

where

$$\Phi_1 = K_1 \log \rho_1, \quad \rho_1 = \frac{r_1}{\alpha}, \quad \Phi_2 = K_2 \log \rho_2, \quad \rho_2 = \frac{r_2}{1 - \alpha} = \frac{-b_1 + b_2 + K_1 \rho_1}{K_2}, \quad (21)$$

with α given by (15). The system is now a system of four scalar PDEs with the four unknowns r_1, r_2, u_1 and u_2 .

3. Statement of the weak asymptotic method.

Setting

$$u_i^+ = \frac{|u_i| + u_i}{2}, \quad u_i^- = \frac{|u_i| - u_i}{2}, \quad (22)$$

one has

$$u_i^+ - u_i^- = u_i, \quad u_i^+ + u_i^- = |u_i|. \quad (23)$$

The two continuity equations and the two Euler equations are replaced by the following ODEs, $i = 1, 2$

$$\frac{d}{dt} r_i(x, t, \epsilon) = \frac{1}{\epsilon} [(r_i u_i^+)(x - \epsilon, t, \epsilon) - (r_i |u_i|)(x, t, \epsilon) + (r_i u_i^-)(x + \epsilon, t, \epsilon)] + \epsilon^\beta, \quad (24)$$

with $\beta > 0$ to be defined later,

$$\begin{aligned} \frac{d}{dt} (r_i u_i)(x, t, \epsilon) &= \frac{1}{\epsilon} [(r_i u_i u_i^+)(x - \epsilon, t, \epsilon) - \\ & (r_i u_i |u_i|)(x, t, \epsilon) + (r_i u_i u_i^-)(x + \epsilon, t, \epsilon)] - r_i(x, t, \epsilon) \frac{\partial}{\partial x} \Phi_i(x, t, \epsilon) + g r_i(x, t, \epsilon). \end{aligned} \quad (25)$$

The potentials Φ_i , $i = 1, 2$, are defined by

$$\Phi_i(x, t, \epsilon) = K_i [\log(\rho_i(\cdot, t, \epsilon) + \epsilon^N) * \phi_{\epsilon^\gamma}](x), \quad (26)$$

with N and γ to be defined later, $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\phi \geq 0$ and $\int \phi(\mu) d\mu = 1$. The convolution in (26) permits that the derivative $\frac{\partial}{\partial x} \Phi_i$ in (25) makes sense: thus the fact the equations (19, 20) are not in divergence form does not cause any trouble for the approximating sequences. We recall α is defined in (15), $\rho_1 = \frac{r_1}{\alpha}$, and one will prove $\alpha_i(x, t, \epsilon) > 0 \forall \epsilon > 0$; ρ_2 is given in (12, 21), $u_i = \frac{r_i u_i}{r_i}$ and one will prove $r_i(x, t, \epsilon) > 0 \forall \epsilon > 0$. This will follow from (33) below, which, from (13), implies $\alpha \neq 0$ and $\alpha \neq 1$.

We assume $r_{i,0}$ and $u_{i,0}$, $i = 1, 2$ are given initial conditions on the 1-D torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with the properties $r_{i,0} \in L^1(\mathbb{T})$ and $u_{i,0} \in L^\infty(\mathbb{T})$ and that $r_{i,0}^\epsilon$ and $u_{i,0}^\epsilon$ are regularizations of $r_{i,0}$ and $u_{i,0}$ respectively, with uniform L^1 and L^∞ bounds respectively (independent on ϵ), and $r_{i,0}^\epsilon(x) > 0 \forall x$.

Theorem. *If $0 < \gamma < \frac{1}{6}$ and $N - 1 - \beta - 3\gamma > 0$ the system of four ODEs (24, 25) complemented by the relations (6, 7, 26) provides a weak asymptotic method for the system (17, 18, 19, 20, 21).*

Sections 4 and 5 are devoted to the proof of the Theorem.

4. A priori inequalities for fixed ϵ .

We seek solutions on the 1-D torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Families $(r_{i,0}^\epsilon)_\epsilon$ and $(r_{i,0}^\epsilon u_{i,0}^\epsilon)_\epsilon$ of approximations of initial conditions are given on \mathbb{T} . For fixed

$\epsilon > 0$ we assume existence and uniqueness of a solution of (24, 25) of class \mathcal{C}^1

$$\begin{aligned} [0, \delta(\epsilon)[&\longmapsto \mathcal{C}_b(\mathbb{R})^4, \\ t &\longmapsto (r_i(\cdot, t, \epsilon), r_i u_i(\cdot, t, \epsilon)) \end{aligned}$$

such that

$$\exists m > 0 / r_i(x, t, \epsilon) \geq m \quad \forall x \in \mathbb{R} \quad \forall t \in [0, \delta(\epsilon)[, \quad (27)$$

$$\exists M > 0 / \|u_i(\cdot, t, \epsilon)\|_\infty \leq M, \|r_i(\cdot, t, \epsilon)\|_\infty \leq M \quad \forall t \in [0, \delta(\epsilon)[. \quad (28)$$

Proposition 1 (a priori inequalities).

- $\forall \epsilon > 0, \quad \forall t \in [0, \delta(\epsilon)[\quad r_i(\cdot, t, \epsilon) \in L^1(\mathbb{T})$, and

$$\int_{-\pi}^{\pi} r_i(x, t, \epsilon) dx = \int_{-\pi}^{\pi} r_i(x, 0, \epsilon) dx + 2\pi\epsilon^\beta t, \quad (29)$$

- $\exists C > 0 / \quad \left\| \frac{\partial}{\partial x} \Phi_i(\cdot, t, \epsilon) \right\|_\infty \leq \frac{C}{\epsilon^{3\gamma}} \quad \forall t \in [0, \delta(\epsilon)[, \forall \epsilon > 0,$ (30)

- $\|u_i(\cdot, t, \epsilon)\|_\infty \leq \|u_i(\cdot, 0, \epsilon)\|_\infty + \frac{2(C+g)}{\epsilon^{3\gamma}} \delta(\epsilon) \quad \forall t \in [0, \delta(\epsilon)[, \forall \epsilon > 0.$ (31)

Setting

$$k(\epsilon) = \max_{i=1,2} \|u_i(\cdot, 0, \epsilon)\|_\infty + \frac{2(C+g)\delta(\epsilon)}{\epsilon^{3\gamma}}, \quad (32)$$

then $\forall t \in [0, \delta(\epsilon)[, \forall \epsilon > 0$,

- $r_i(x, 0, \epsilon) \exp\left(\frac{-k(\epsilon)t}{\epsilon}\right) \leq r_i(x, t, \epsilon) \leq 2\|r_i(\cdot, 0, \epsilon)\|_\infty \exp\left(\frac{2k(\epsilon)t}{\epsilon}\right) \quad \forall x \in \mathbb{R}.$ (33)

Proof of Proposition 1.

- From (23, 24),

$$\begin{aligned} \frac{d}{dt} \int_{-\pi}^{+\pi} r_i(x, t, \epsilon) dx &= \frac{1}{\epsilon} [\int_{-\pi}^{+\pi} (r_i u_i^+)(x - \epsilon, t, \epsilon) dx - \int_{-\pi}^{+\pi} (r_i u_i^+)(x, t, \epsilon) dx - \\ &\int_{-\pi}^{+\pi} (r_i u_i^-)(x, t, \epsilon) dx + \int_{-\pi}^{+\pi} (r_i u_i^-)(x + \epsilon, t, \epsilon) dx] + 2\pi\epsilon^\beta = 0 + 2\pi\epsilon^\beta \end{aligned}$$

by periodicity of r_i and u_i .

- From (26),

$$\frac{\partial}{\partial x} (\Phi_i)(x, t, \epsilon) = K_i \int \log[\rho_i(x - y, t, \epsilon) + \epsilon^N] \frac{1}{\epsilon^{2\gamma}} \phi'\left(\frac{y}{\epsilon^\gamma}\right) dy.$$

If $\rho_i(x-y, t, \epsilon) \leq 1$, one uses the bound $|\log(\epsilon^N)| \leq \frac{\text{const}}{\epsilon^\gamma}$. If $\rho_i(x-y, t, \epsilon) > 1$, one uses the fact that $\rho_i(\cdot, t, \epsilon) \in L^1(\mathbb{T})$ with L^1 norm independent on ϵ and $t \in [0, \delta(\epsilon)[$. The result that $\rho_i(\cdot, t, \epsilon) \in L^1(\mathbb{T})$ with L^1 norm independent on ϵ and t follows from formula (16) that implies $\rho_1 = \frac{r_1}{\alpha} \leq r_1 \frac{b_1 - b_2 + K_1 r_1 + K_2 r_2}{K_1 r_1}$. Then one notices that $b_1 - b_2 > 0, K_i > 0, r_i > 0$ and the result follows from (29). For ρ_2 one uses (12).

• Now we proceed to the proof of (31). From (24) and the assumption that the solution of the ODEs is of class \mathcal{C}^1 on $[0, \delta(\epsilon)[$ valued in the Banach space $\mathcal{C}(\mathbb{T})$, one obtains, for fixed $\epsilon > 0$ and for $dt > 0$ small enough with $t + dt < \delta(\epsilon)$, that

$$\begin{aligned} r_i(x, t + dt, \epsilon) &= r_i(x, t, \epsilon) + \\ \frac{dt}{\epsilon} [(r_i u_i^+)(x - \epsilon, t, \epsilon) - (r_i |u_i|)(x, t, \epsilon) + (r_i u_i^-)(x + \epsilon, t, \epsilon)] + dt \cdot o(x, t, \epsilon)(dt) + \epsilon^\beta dt &= \\ \frac{dt}{\epsilon} (r_i u_i^+)(x - \epsilon, t, \epsilon) + (1 - \frac{dt}{\epsilon} |u_i|(x, t, \epsilon)) r_i(x, t, \epsilon) + \frac{dt}{\epsilon} (r_i u_i^-)(x + \epsilon, t, \epsilon) + dt \cdot o(x, t, \epsilon)(dt) + \epsilon^\beta dt & \\ & \quad (34) \end{aligned}$$

where $\|o(\cdot, t, \epsilon)(dt)\|_\infty \rightarrow 0$ when $dt \rightarrow 0$ uniformly for t in a compact set of $[0, \delta(\epsilon)[$, from the mean value theorem in the form $f(t + dt) = f(t) + dt f'(t) + dt \cdot r(t, dt)$, with $\|r(t, dt)\| \leq \sup_{0 < \theta < 1} \|f'(t + \theta dt) - f'(t)\|$. Notice that there is no uniformness in ϵ . For $dt > 0$ small enough (depending on ϵ) the single term $(1 - \frac{dt}{\epsilon} |u_i|(x, t, \epsilon)) r_i(x, t, \epsilon)$ dominates the term $dt \cdot o(x, t, \epsilon)(dt)$ from (27, 28). Since, further, $r_i u_i^\pm \geq 0$, one can invert (34). Dropping the useless term $\epsilon^\beta dt$ one obtains

$$\begin{aligned} \frac{1}{r_i(x, t + dt, \epsilon)} &\leq \\ [\frac{dt}{\epsilon} (r_i u_i^+)(x - \epsilon, t, \epsilon) + [1 - \frac{dt}{\epsilon} |u_i|(x, t, \epsilon)] r_i(x, t, \epsilon) + \frac{dt}{\epsilon} (r_i u_i^-)(x + \epsilon, t, \epsilon)]^{-1} + \\ dt \cdot o(x, t, \epsilon)(dt) & \end{aligned}$$

where the new o has still the property that $\|o(\cdot, t, \epsilon)(dt)\|_\infty \rightarrow 0$ when $dt \rightarrow 0$ uniformly for $t \in [0, \delta']$ if $\delta' < \delta(\epsilon)$.

Applying the analog of (34) for $r_i u_i$ in place of r_i , with the supplementary terms $r_i \frac{\partial}{\partial x}(\Phi_i)$ and gr_i from (25), one obtains, using (27, 28)

$$u_i(x, t + dt, \epsilon) = \frac{(r_i u_i)(x, t + dt, \epsilon)}{r_i(x, t + dt, \epsilon)} \leq$$

$$\begin{aligned} & \frac{\frac{dt}{\epsilon}(r_i u_i u_i^+)(x - \epsilon, t, \epsilon) + [1 - \frac{dt}{\epsilon}|u_i|(x, t, \epsilon)](r_i u_i)(x, t, \epsilon) + \frac{dt}{\epsilon}(r_i u_i u_i^-)(x + \epsilon, t, \epsilon)}{\frac{dt}{\epsilon}(r_i u_i^+)(x - \epsilon, t, \epsilon) + [1 - \frac{dt}{\epsilon}|u_i|(x, t, \epsilon)]r_i(x, t, \epsilon) + \frac{dt}{\epsilon}(r_i u_i^-)(x + \epsilon, t, \epsilon)} \\ & + dt \frac{r_i(x, t, \epsilon)}{r_i(x, t + dt, \epsilon)} \left[\left| \frac{\partial}{\partial x}(\Phi_i)(x, t, \epsilon) \right| + g \right] + dt \cdot o(x, t, \epsilon)(dt) \end{aligned} \quad (35)$$

where the new o has the same property as in (34) for fixed ϵ . For $dt > 0$ small enough the first term in the second member is a barycentric combination of $u_i(x - \epsilon, t, \epsilon)$, $u_i(x, t, \epsilon)$ and $u_i(x + \epsilon, t, \epsilon)$. The quotient $\frac{r_i(x, t + dt, \epsilon)}{r_i(x, t, \epsilon)}$ tends to 1 when $dt \rightarrow 0$ (for fixed ϵ). Finally one obtains, using also (30), that

$$\|u_i(\cdot, t + dt, \epsilon)\|_\infty \leq \|u_i(\cdot, t, \epsilon)\|_\infty + dt \frac{const}{\epsilon^{3\gamma}} + dt \cdot \|o(\cdot, t, \epsilon)(dt)\|_\infty \quad (36)$$

with uniform bound of o when t ranges in a compact set in $[0, \delta(\epsilon)[$, for fixed ϵ . One obtains the bound (31) as in [6] by dividing the interval $[0, t]$ into n small intervals $[\frac{it}{n}, \frac{(i+1)t}{n}]$, $0 \leq i \leq n - 1$, applying (36) in each small interval, which gives

$$\|u_i(\cdot, (i+1)\frac{t}{n}, \epsilon)\|_\infty \leq \|u_i(\cdot, i\frac{t}{n}, \epsilon)\|_\infty + \frac{t}{n} \frac{const}{\epsilon^{3\gamma}} + \frac{t}{n} o(\frac{t}{n}),$$

summing on i and using that $o(\frac{t}{n}) \rightarrow 0$ when $n \rightarrow \infty$.

•The proofs of the two inequalities (33) follows from (24) that gives the inequalities $\frac{d}{dt}r_i(x, t, \epsilon) \geq -\frac{\|u_i\|_\infty}{\epsilon}r_i(x, t, \epsilon)$ and $\frac{d}{dt}r_i(x, t, \epsilon) \leq \frac{2\|u_i\|_\infty\|r_i\|_\infty}{\epsilon}$ using (31) to evaluate $\|u_i\|_\infty$. They are given in detail in section 2 of [6].

The existence of a unique global solution to (24, 25) for fixed ϵ is obtained from the a priori estimates (29-33) from classical ODEs arguments of the theory of ODEs in Banach spaces in the Lipschitz case. Indeed for fixed $\epsilon > 0$, if $0 < \lambda < 1$ and $\Omega_\lambda := \{(X_i, Y_i) \in \mathcal{C}(\mathbb{T})^4 / \forall x \in \mathbb{T} \lambda < X_i(x) < \frac{1}{\lambda}, |Y_i(x)| < \frac{1}{\lambda}\}$ the four equations (24, 25) with variables $X_i = r_i$ and $Y_i = r_i u_i$ have the Lipschitz property on Ω_λ with values in $\mathcal{C}(\mathbb{T})^4$, with Lipschitz constants $\leq \frac{1}{\lambda^3}$. We refer to [6] section 4 for details.

5. Proof of the weak asymptotic method.

It remains to prove that the solution of the system of ODEs (24, 25) and the formula (26) provide a weak asymptotic method for system (17, 18, 19, 20, 21) when $\epsilon \rightarrow 0$. To this end one has to prove that $\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R})$, (37-39) below hold when $\epsilon \rightarrow 0$

$$\int \frac{d}{dt}r_i(x, t, \epsilon)\psi(x)dx - \int (r_i u_i)(x, t, \epsilon)\psi'(x)dx \rightarrow 0, \quad (37)$$

$$\int \frac{d}{dt}(r_i u_i)(x, t, \epsilon) \psi(x) dx - \int (r_i(u_i)^2)(x, t, \epsilon) \psi'(x) dx + \int r_i(x, t, \epsilon) \frac{\partial}{\partial x}(\Phi_i)(x, t, \epsilon) \psi(x) dx - g \int r_i(x, t, \epsilon) \psi(x) dx \rightarrow 0, \quad (38)$$

$$\int \Phi_i(x, t, \epsilon) \psi(x) dx - K_i \int \log[\rho_i(x, t, \epsilon)] \psi(x) dx \rightarrow 0 \quad (39)$$

where (37) means satisfaction of (17, 18), (38) satisfaction of (19, 20) and (39) satisfaction of the two state laws in (21) in the sense of distributions at the limit $\epsilon \rightarrow 0$.

The proof of (37) is as follows: from (23, 24, 29, 30, 31), a change of variable and $\frac{\psi(x+\epsilon)-\psi(x)}{\epsilon} = \psi'(x) + O_x(\epsilon)$,

$$\begin{aligned} \int \frac{d}{dt} r_i(x, t, \epsilon) \psi(x) dx &= \frac{1}{\epsilon} \int (r_i u_i^+)(x, t, \epsilon) [\psi(x+\epsilon) - \psi(x)] dx - \frac{1}{\epsilon} \int (r_i u_i^-)(x, t, \epsilon) \\ &[\psi(x) - \psi(x-\epsilon)] dx + \int \epsilon^\beta \psi(x) dx = \int (r_i u_i)(x, t, \epsilon) \psi'(x) dx + \int_{compact} (r_i u_i^+)(x, t, \epsilon) \\ &O_x(\epsilon) dx + \int_{compact} (r_i u_i^-)(x, t, \epsilon) O_x(\epsilon) dx + O(\epsilon^\beta) = \int (r_i u_i)(x, t, \epsilon) \psi'(x) dx + \\ &(const + \frac{const}{\epsilon^{3\gamma}}) O(\epsilon) + O(\epsilon^\beta) = \int (\rho u)(x, t, \epsilon) \psi'(x) dx + O(\epsilon^{1-3\gamma}) + O(\epsilon^\beta). \end{aligned}$$

This gives (37) if $0 < \gamma < \frac{1}{3}$. The proof of (38) is similar since the additional terms $\int r_i(x, t, \epsilon) \frac{\partial}{\partial x} \Phi_i(x, t, \epsilon) \psi(x) dx$ and $g \int r_i(x, t, \epsilon) \psi(x) dx$ are the same in (25) and (38): one obtains a remainder $\frac{const}{\epsilon^{6\gamma}} O(\epsilon)$ because of one more factor u_i and the bound (31). Finally one chooses $0 < \gamma < \frac{1}{6}$.

To check (39) one has to prove from (26) that $\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$\int \{[(\log(\rho_i(\cdot, t, \epsilon) + \epsilon^N) * \phi_{\epsilon^\gamma})(x) - \log[\rho_i(x, t, \epsilon)]]\} \psi(x) dx \rightarrow 0 \quad (40)$$

when $\epsilon \rightarrow 0$. To this end we share the integral (40) into two parts (41, 42) below and we prove that each tends to 0 when $\epsilon \rightarrow 0$. Let

$$I = \int \{[(\log(\rho_i(\cdot, t, \epsilon) + \epsilon^N) * \phi_{\epsilon^\gamma})(x) - \log[\rho_i(x, t, \epsilon) + \epsilon^N]]\} \psi(x) dx \quad (41)$$

and

$$J = \int \{(\log[\rho_i(x, t, \epsilon) + \epsilon^N] - \log[\rho_i(x, t, \epsilon)])\} \psi(x) dx. \quad (42)$$

Now

$$I = \int \{(\log[\rho_i(x - \epsilon^\gamma \mu, t, \epsilon) + \epsilon^N] - \log[\rho_i(x, t, \epsilon) + \epsilon^N])\} \phi(\mu) \psi(x) d\mu dx = \\ \int \log[\rho_i(x, t, \epsilon) + \epsilon^N] \phi(\mu) [\psi(x + \epsilon^\gamma \mu) - \psi(x)] d\mu dx.$$

Since $\rho_i(x, t, \epsilon) \geq 0$ from (16, 21, 33), using its L^1 property (29) in the case $\rho_i(x, t, \epsilon) > 1$ and using the term ϵ^N in the case $\rho_i(x, t, \epsilon) \leq 1$, as in the proof of (30), one has $|I| \leq \text{const.} \log(\frac{1}{\epsilon}) \epsilon^\gamma$. Therefore $I \rightarrow 0$ when $\epsilon \rightarrow 0$.

Now, (42) and the mean value theorem give

$$|J| \leq \epsilon^N \frac{1}{\min(\rho_i)} \text{const} \quad (43)$$

if $\min(\rho_i)$ denotes the inf of $\rho_i(x, t, \epsilon)$ for fixed t, ϵ when x ranges in a compact set containing the support of ψ . The problem is to obtain an inf. bound of $\min(\rho_i)$; this is the purpose of the term ϵ^β in (24). From (24), $\frac{dr_i}{dt}(x, t, \epsilon) \geq -\frac{1}{\epsilon} r_i(x, t, \epsilon) \|u_i(\cdot, t, \epsilon)\|_\infty + \epsilon^\beta \geq -\frac{1}{\epsilon} r_i(x, t, \epsilon) \frac{\text{const}}{\epsilon^{3\gamma}} T + \epsilon^\beta$ if $t \in [0, T]$, from (31) applied with $\delta(\epsilon) = T$.

Setting $A := \text{const} \frac{T}{\epsilon^{1+3\gamma}}$ and $B := \epsilon^\beta$, one has $\frac{dr_i}{dt} \geq -Ar_i + B$. Comparing with the exact solution of the ODE $\frac{dX}{dt}(x, t) = -AX(x, t) + B$ with initial condition $X(x, 0) = r_{i,0}(x, \epsilon)$, namely $X(x, t) = r_{i,0}(x, \epsilon) e^{-At} + \frac{B}{A}(1 - e^{-At}) \geq \frac{B}{A}(1 - e^{-At})$, we obtain the bound

$$r_i(x, t, \epsilon) \geq \text{const}(t) \cdot \epsilon^{1+\beta+3\gamma} \quad (44)$$

for $\epsilon > 0$ small enough and fixed t . Now using (16) we can obtain a lower bound of $\min \rho_i$

$$\rho_1(x, t, \epsilon) = \frac{r_1(x, t, \epsilon)}{\alpha(x, t, \epsilon)} \geq r_1(x, t, \epsilon) \geq \text{const.} \epsilon^{1+\beta+3\gamma}. \quad (45)$$

Similarly, from (12)

$$\rho_2(x, t, \epsilon) \geq \text{const.} \epsilon^{1+\beta+3\gamma}. \quad (46)$$

From (43), $|J| \leq \text{const}(t) \cdot \epsilon^{N-1-\beta-3\gamma}$ and it suffices to choose $N - 1 - \beta - 3\gamma > 0$ to obtain that $J \rightarrow 0$ when $\epsilon \rightarrow 0$. \square

6. Numerical observations from the weak asymptotic method.

We will present two shock tube problems selected from [14]. The pressure laws are those in [14] p. 179-180: $K_1 = 10^6$, $K_2 = 10^5$, $b_i = K_i \rho_{0,i} - p_{0,i}$, $\rho_{0,1} = 1000$, $p_{0,1} = 10^5$, $\rho_{0,2} = 0$ and $p_{0,2} = 0$. The final time is $T = 0.001$, with 1000

cells on $[0,1]$ (or $T = 0.01$ on $[0,10]$), therefore $\Delta x = \epsilon = (1000)^{-1}$ and the CFL number is $r = \frac{\Delta t}{\Delta x} = 10^{-6}$. We use the explicit Euler order one method for the ODEs (24, 25). We choose $\delta = 1, \beta = 100$ and $N = 100$ (β and N do not matter since there is no void region in any fluid). We regularize the initial conditions $\omega_{i,0} = r_{i,0}, r_{i,0}u_{i,0}$ by an averaging

$$\nu\omega_{i,0}(x - \epsilon) + (1 - 2\nu)\omega_{i,0}(x) + \nu\omega_{i,0}(x + \epsilon), \quad \nu = 0.1. \quad (47)$$

We represent the convolution in (26) by a similar averaging of Φ_i on 5 cells instead of 3 with coefficient $\nu=0.15$. We use a small averaging as (47) at each step in r_i and $r_i u_i, i = 1, 2$. Concerning this last averaging one observes that the minimal needed values of ν tend to 0 when $r \rightarrow 0$: $\nu = 10^{-2}, 10^{-3}$ and 10^{-4} when $r = 10^{-4}, 10^{-5}$ and 10^{-6} respectively. Therefore this regularization can be considered as a numerical artefact absent in the ODE formulation which corresponds to $r = 0$. The Riemann conditions are $\alpha = 0.71, 0.7, p = 265000, 265000, u_1 = 1, 1$ and $u_2 = 65, 50$ for test 1 and $\alpha = 0.7, 0.1, p = 265000, 265000, u_1 = 10, 15$ and $u_2 = 65, 50$ for test 2.

Observation 1. For shock tube problem 1 (figure 1) one observes the same results as those depicted in [14]. For shock tube problem 2 (figure 2) one observes a slight difference for the second step value in the right panels: $2.46 \cdot 10^5$ instead of $2.50 \cdot 10^5$ (top panel) and 89 instead of 84 (bottom panel). These values do not change with discretizations ranging from 100 to 20000 cells, with different values of r and the other parameters, and are also exactly those obtained from the direct adaptation of the scheme in section 7 below. With the pressure correction adopted in [14] one observes from the scheme in section 7 that this difference tends to disappear, figures 4 and 5, therefore it is presumably a consequence of the pressure correction adopted in [14]. Modulo this difference one observes that the results we obtain without pressure correction agree with the results obtained in [14] even with pressure correction, which appear therefore as depictions of approximate solutions.

Observation 2. In both tests it has been observed that the left and right discontinuities satisfy with great precision the 3 standard jump conditions of system (8-11): the two ones from (8, 9) and the third one from the equation obtained by adding (10) and (11). They satisfy also with great precision the two formal jump conditions (62) one can calculate from nonlinear algebraic calculations with the nonconservative equations as done in the appendix.

The arrays below give on a line the values of the wave velocities computed from the two equations (8, 9) i.e. $c = \frac{[r_1 u_1]}{[r_1]}$ and $c = \frac{[r_2 u_2]}{[r_2]}$, the value computed adding the equations (10, 11) without gravitation, i.e. $c = \frac{[r_1(u_1)^2 + r_2(u_2)^2 + p]}{[r_1 u_1 + r_2 u_2]}$,

and the two formal results (62). They are calculated from the numerical step values in figures 1 and 2. We first give the results for the shock tube problem 1, then for the shock tube problem 2. When the jump conditions are satisfied all values on a line should be equal since they are the value of the velocity of a shock wave obtained from the 5 different formulas.

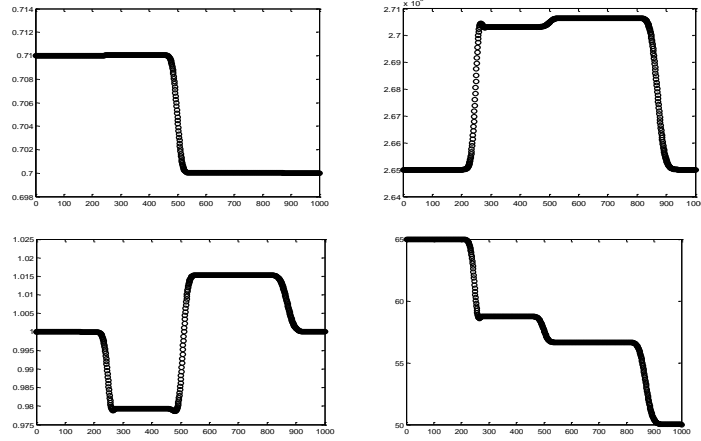


Figure 1. Shock tube problem 1. Numerical solution of the ODEs of the weak asymptotic method. From top left to bottom right: liquid fraction, pressure, liquid velocity, gas velocity.

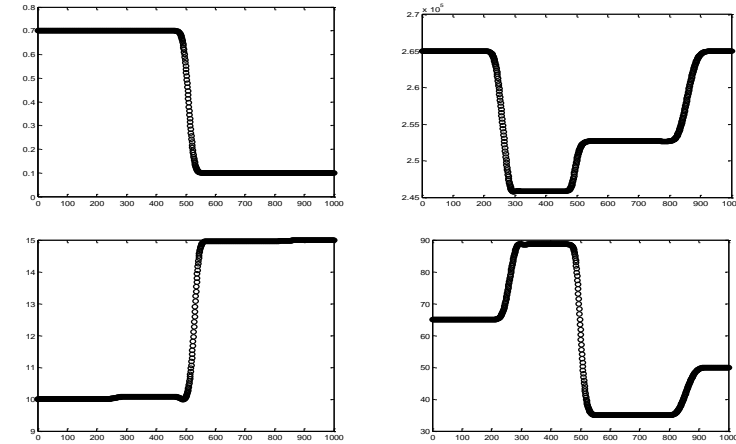


Figure 2. Shock tube problem 2. Numerical solution of the ODEs of the weak asymptotic method. From top left to bottom right: liquid fraction, pressure, liquid velocity, gas velocity.

shock tube problem 1					
	c_1	c_2	c_3	c_4	c_5
left	-255.95	-255.90	-255.82	-255.84	-255.75
middle	-1.52	-1.49	17.94	10.23	-0.95
right	370.35	370.23	370.24	370.28	370.15

shock tube problem 2					
	c_1	c_2	c_3	c_4	c_5
left	-240.23	-240.85	-240.81	-240.65	-240.91
middle	9.34	9.30	8.30	13.78	10.52
right	358.32	358.70	358.84	358.53	358.77

These jump formulas are very well satisfied by the left and right discontinuities but are not satisfied by the middle discontinuity except the two jump conditions from the two continuity equations. Since it is proved the results depict approximate solutions (from the theorem and from a careful numerical solution of the ODEs) an explanation could be that the middle discontinuity is not a classical shock wave as suggested by the singularities often observed on top or bottom of this discontinuity, which could denote it is some kind of more complicated wave, possibly not a shock wave. To test this hypothesis we did numerical tests for different volumic compositions of the fluids. One observes that in the case of equal volume fractions on both sides in the Riemann problem there appears a very neat singularity in the middle discontinuity which is present in the other cases but far clearly visible on the volumic fraction when both sides of the volumic fraction are equal (figure 3). With the values of pressure and velocities of shock tube problem 1 the observed singularity in volumic fraction is small, while it is quite large with the values of shock wave problem 2, figure 3. This explains why the middle discontinuity does not satisfy well, or does not satisfy at all in some cases, the expected conservative jump conditions: it is not a classical shock wave i.e. a mere moving discontinuity. It is natural that something else than a mere shock wave occurs: if the Riemann problem were solved by three standard shock waves we would have $8+3=11$ unknown values (the 3 velocities and the 8 step values) for 12 equations (the 4 jump conditions at each discontinuity supposing one has solved the ambiguity in the 2 nonconservative equations). The values of wave velocities in case of figure 3, computed from the 5 algebraic formulas as in the above arrays are

shock tube problem 3					
	c_1	c_2	c_3	c_4	c_5
left	-253.35	-253.34	-253.33	-253.33	-253.31
middle	-8473	-1099	25.3	12.8	-16.9
right	368.99	368.96	368.96	368.97	368.92

Since we have an approximate solution that can be computed with arbitrary precision it is possible to observe this singular part of the solution. Numerical

investigation on the "object" that appears in the liquid fraction for $\alpha = 0.60$ (top-left panel in figure 3) shows that

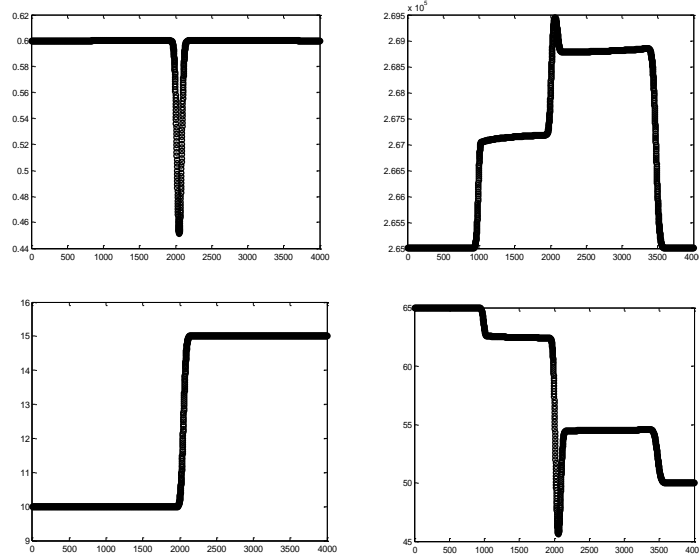


Figure 3. An approximate solution from the weak asymptotic method for the Riemann problem obtained from shock tube problem 2 by choosing the initial volumic fraction of liquid =0.6 on both sides and using 4000 space steps. One observes the middle singularities in liquid fraction (top left panel), in pressure (top right panel) and in gas velocity (bottom right panel).

- The area of the region located below the line $\alpha = 0.60$ and above the curve α is constant (independent on ϵ) for fixed time when ϵ varies and it is proportional to time even up to very large values of time (tests were done up to 100 times the value of time used in figures 1, 2 and 3 with the scheme in section 7).

- For rather small values of the time such as those in figures 1, 2 and 3 the object travels with constant speed and its width on $\alpha = 0.60$ tends to 0 when $\epsilon \rightarrow 0$, roughly as $\sqrt{\epsilon}$ for fixed t and as \sqrt{t} for fixed ϵ ; its minimum value diminishes when the time increases. For large values of the time this decrease of the minimum is stopped because one always has $\alpha(x) > 0 \forall x$ and one observes the width of the object then increases proportionally to t so as to maintain an area proportional to time.

Observation 3. One could state different values of γ for the spreading of the state laws of the two fluids when they are very different, for instance a liquid and a gas: we have observed that modifications representing the convolution are unefficient to produce significative differences in the solution

because in the present case the discontinuities take place on a large number of cells thus making the results rather unsensitive to modifications that would not be important enough to modify significantly the aspect of the jumps of α and p (the nonconservative terms in (10, 11)): indeed the great sensibility on the slight modifications of the schemes for systems in nondivergence form has been observed in the case the discontinuities take place on a very small number of cells. The numerical schemes observed in the case of the multifluid system are robust in the sense that small modifications of the scheme do not affect significantly the result precisely because the discontinuities are spread over a large number of cells. Because of this fact one observes in the three arrays corresponding to figures 1, 2 and 3 that not only the conservative jump conditions (the three values c_1, c_2 and c_3) but also the two formal jump conditions (the two values c_4 and c_5) are satisfied, showing the evidence that, to some extent, one can compute formally on the system, thus allowing the formal nonlinear calculation done to obtain formula (13).

7. A transport-correction scheme.

We propose here a natural numerical scheme for the numerical solution of system (17-21). The scheme is an adaptation of the Le Roux et al numerical method of splitting into transport and pressure correction as described in [3], extending to two fluids the scheme done in [5] for one fluid. The space $\mathbb{R} \times [0, +\infty[$ is divided into rectangular cells $[ih - \frac{h}{2}, ih + \frac{h}{2}] \times [n\Delta t, (n+1)\Delta t[, i \in \mathbb{Z}, n \in \mathbb{N}$.

Given the family $\{(r_1)_i^n, (r_2)_i^n, (r_1 u_1)_i^n, (r_2 u_2)_i^n\}_{i \in \mathbb{Z}}$ of values of these variables on the interval $[ih - \frac{h}{2}, ih + \frac{h}{2}]$ at time $n\Delta t$ we seek the family of values $\{(r_1)_i^{n+1}, (r_2)_i^{n+1}, (r_1 u_1)_i^{n+1}, (r_2 u_2)_i^{n+1}\}_{i \in \mathbb{Z}}$ at time $(n+1)\Delta t$.

- *First step: transport.* For $k=1, 2$

$$(u_k)_i^n := \frac{(r_k u_k)_i^n}{(r_k)_i^n} \quad (48)$$

if $(r_k)_i^n \neq 0$, any value if $(r_k)_i^n = 0$,

$$(u_k)_i^{n,+} := \frac{|(u_k)_i^n| + (u_k)_i^n}{2}, (u_k)_i^{n,-} := \frac{|(u_k)_i^n| - (u_k)_i^n}{2}. \quad (49)$$

The CFL condition is $r|(u_k)_i^n| < 1 \forall k, i, n$. Then if $r = \frac{\Delta t}{h}$ we set

$$(\overline{r_k})_i := r(r_k)_{i-1}^n (u_k)_{i-1}^{n,+} + (1 - r|(u_k)_i^n|)(r_k)_i^n + r(r_k)_{i+1}^n (u_k)_{i+1}^{n,-}, \quad (50)$$

$$(\overline{r_k u_k})_i := r(r_k u_k)_{i-1}^n (u_k)_{i-1}^{n,+} + (1 - r|(u_k)_i^n|)(r_k u_k)_i^n + r(r_k u_k)_{i+1}^n (u_k)_{i+1}^{n,-}. \quad (51)$$

- *Second step: averaging.* We choose a value $\mu, 0 < \mu < 0.5$,

$$(r_k)_i^{n+1} := \mu(\overline{r_k})_{i-1} + (1 - 2\mu)(\overline{r_k})_i + \mu(\overline{r_k})_{i+1}, \quad (52)$$

$$\widetilde{(r_k u_k)_i} := \mu(\overline{r_k u_k})_{i-1} + (1 - 2\mu)(\overline{r_k u_k})_i + \mu(\overline{r_k u_k})_{i+1}. \quad (53)$$

- *Third step: pressure correction.*

$$\Delta_i := (K_1(\overline{r_1})_i + K_2(\overline{r_2})_i + b_1 - b_2)^2 - 4(b_1 - b_2)K_1(\overline{r_1})_i, \quad (54)$$

$$\alpha_i := \frac{K_1(\overline{r_1})_i + K_2(\overline{r_2})_i + b_1 - b_2 - \sqrt{(\Delta)_i}}{2(b_1 - b_2)}, \quad (55)$$

$$p_i := K_1 \frac{(\overline{r_1})_i}{\alpha_i} - b_1 \quad \text{if } \alpha_i \neq 0, \quad (56)$$

$$(r_1 u_1)_i^{n+1} := \widetilde{(r_1 u_1)_i} - \frac{r}{2} \alpha_i (p_{i+1} - p_{i-1}), \quad (57)$$

$$(r_2 u_2)_i^{n+1} := \widetilde{(r_2 u_2)_i} - \frac{r}{2} (1 - \alpha_i) (p_{i+1} - p_{i-1}). \quad (58)$$

In (52, 58) we have obtained the family $\{(r_1)_i^{n+1}, (r_2)_i^{n+1}, (r_1 u_1)_i^{n+1}, (r_2 u_2)_i^{n+1}\}_{i \in \mathbb{Z}}$.

Now we justify the choice of an arbitrary value in density when a denominator in (48) is null.

Proposition. *When $(r_k)_i^{n+1} = 0, k = 1$ or 2 , then $(r_k u_k)_i^{n+1} = 0$.*

proof. Assume $(r_k)_i^{n+1} = 0$. Then from (52), the strict inequality in μ and the positiveness of r_k imply

$$(\overline{r_k})_{i-1} = 0 = (\overline{r_k})_i = (\overline{r_k})_{i+1}. \quad (59)$$

Now notice that $(\overline{r_k})_i = 0$ implies $(r_k)_i^n = 0$ from (50) and the strict inequality in the CFL condition. Further since $r \neq 0$ it also implies from (50)

that $(r_k)_{i-1}^n (u_k)_{i-1}^{n,+} = 0$, which implies $(r_k u_k)_{i-1}^n (u_k)_{i-1}^{n,+} = 0$, and similarly $(r_k u_k)_{i+1}^n (u_k)_{i+1}^{n,-} = 0$. Therefore $(\bar{r}_k)_i = 0$ implies $(\bar{r}_k u_k)_i = 0$. Therefore from (59)

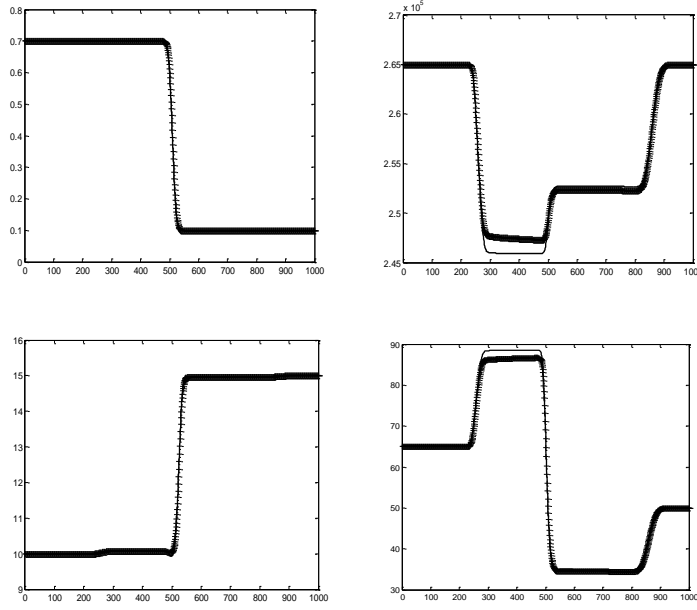


Figure 4. The shock tube problem 2 without correction (continuous curve) and with correction (+).

One observes a small difference in two step values in the right panels.

Figure 4. The shock tube problem 2 without correction (continuous curve) and with correction (+). One observes a small difference in two step values in the right panels.

$$(\bar{r}_k u_k)_{i-1} = 0 = (\bar{r}_k u_k)_i = (\bar{r}_k u_k)_{i+1}. \quad (60)$$

Therefore from (53) $(\widetilde{r_k u_k})_i = 0$. From (56, 59) one has also $p_{i-1} = b_1 = p_i = p_{i+1}$ if $k = 1$. Finally, from (57, 58), we obtain $(r_k u_k)_i^{n+1} = 0$. \square

Following calculations in [4, 5] one can prove, under assumptions to be checked, such as boundedness of the velocity field when $h \rightarrow 0$, that the scheme tends to satisfy the equations when $h \rightarrow 0$.

Numerical observations. First it has been observed that the scheme has always given the same result as the weak asymptotic method. It has the advantage to be more efficient and of a very easy use since one has only to fix the value of the CFL number r and then the value of the averaging parameter μ in (52, 53).

The scheme in this section has been used with the interface pressure modelling (11) in [14] p. 180 which ensures the hyperbolicity of the system. In the case of shock tube problem 2 one can observe a slight difference relatively to the absence of correction (figure 4: 1000 space steps, $r = 0.002$, $\mu = 0.1$):

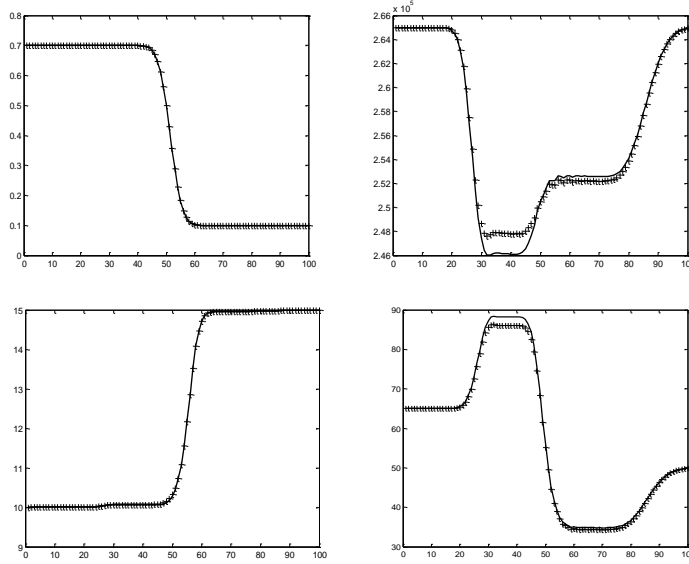


Figure 5. Quality of the transport-correction scheme: +++ results with pressure correction and (continuous line) without pressure correction. The curves are obtained with 100 space steps only.

Figure 5. Quality of the transport-correction scheme: +++ results with pressure correction and (continuous line) without pressure correction. The curves are obtained with 100 space steps only.

the second step values from the left in pressure and gas velocity are 248000 and 86.5 respectively instead of 246000 and 88.5. Since the tests in [14] have been done in presence of this pressure correction and are close to the values we obtain with this correction, this explains the small disagreement observed when comparing the results in [14] figures 4 and 5 p. 197 and 198 with those in figure 2 for these two step values. Besides this difference the results in figure 1 and 3 are unchanged in absence or presence of the pressure correction, in particular the presence of the middle "singular wave" is independent of the presence of pressure correction.

The numerical quality of the scheme is tested in figure 5, both in absence and presence of pressure correction: a discretization in 100 space cells suffices to obtain the step values and the jump formulas (as in the above arrays corresponding to figures 1, 2 and 3) with precision.

9. Conclusion.

The approximate solutions we have constructed with full proof and rather arbitrary initial data provide a mathematical tool that permits theoretical and numerical investigations of the initial value problem for the equal pressure model of multifluid flows in the isothermal case. Since numerical calculations of these approximate solutions can be done easily and accurately with standard ODEs methods these approximate solutions can play the role of explicit solutions for mathematical and numerical investigations. They show that numerical schemes from scientific computing give an approximate solution besides the mathematical peculiarities of the model. They can show that supplementary terms such as pressure corrections do not modify (shock tube problem 1) or modify only slightly (shock tube problem 2) the solution. They permit to investigate the nature of the "solutions" put in evidence by these approximate solutions and by scientific computing although this system is in nondivergence form.

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Appendix. Formal calculations on the system.

We obtain jump formulas from formal calculations. We observe in section 6 that these jump formulas are satisfied by the left shock waves and by the right shock waves in figures 1, 2 and 3. Developping (10) with $g = 0$ and simplifying from (8), then dividing by r_1 one obtains

$$\frac{\partial}{\partial t}(u_1) + u_1 \frac{\partial}{\partial x}(u_1) + \frac{\alpha \frac{\partial}{\partial x} p}{r_1} = 0. \quad (61)$$

Using the state law (6) $p = K_1 \rho_1 - b_1$ and $\rho_1 = \frac{r_1}{\alpha}$ one obtains

$$\frac{\partial}{\partial t}(u_1) = \frac{\partial}{\partial x} \left(-K_1 \log(\rho_1) - \frac{(u_1)^2}{2} \right),$$

which gives the jump condition

$$c = K_1 \frac{\log(\rho_{1,r}) - \log(\rho_{1,l})}{[u_1]} + \frac{u_{1,r} + u_{1,l}}{2} \quad (62)$$

where c denotes the velocity of the shock wave. The same calculation holds from (10) and (9) and gives (62) with index 2 and the same formula with index 2.

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